Dr Satish Kumar, Associate professor
Department of maths D.N. college, Meerut

Book Name: Advanced Mathematical Methods
Subject code: H-3051
Class: M.Sc. Third Semester

Syllabus

This syllabus has been divided into five units whose details are as under:

Unit I

Inner product of functions. Orthogonal set of functions. Fourier series and their properties. Parseval's identities for fourier series, Fourier theorem, Uniform convergence of Fourier series.

Unit - II

Differentiation of Fourier Series, Integration of fourier series. Solution of ordinary boundary value problems in Fourier series. A slab with faces at prescribed tempreture. A Dirichlet problem (in Cartesian co-ordinates only), a string with prescribed initial velocity, an elastic bar, Application of Fourier Series in Sturm Liouville problems.

Dr. Satish Kumar

M-3051

Unit - III

Definitions of integral equation and their classification. Relation between integral and differential equations, Fredholm integral equations of second kind with separable kernels, Reduction to a system of algebraic equations.

Unit - IV

Eigen values and eigen functions, iterated kernels, iterative scheme for solving Fredholm integral equation of second kind (Neumann Series), Resolvent kernel, iterative scheme to Volterra's integral equation of second kind.

Unit - V

Hilbert schmidt theory, symmetric kernels, Orthonormal systems of functions, Fundamental properties of eigen values and eigen functions for symmetric kernels. Solutions of integral equations by using Hilbert Schmidt theory.

11-2021

Inner product of two functions:

Suppose $f(x)$ and $g(x)$ be any two function defined on a given set then their product is given by

$$\langle f(x), g(x) \rangle = \int_a^b f(x)\, g(x)\, dx.$$

Example. Given $f_1(x) = 4$, $f_2(x) = x^3$, $x \in]-2,2[$

Find inner product of $f_1(x)$ and $f_2(x)$.

Solution.

Required inner product is given by

$$\langle 4, x^3 \rangle = \int_{-2}^{2} 4 \cdot x^3\, dx = \left[x^4\right]_{-2}^{2}$$

$$= 0.$$

Norm of a function.

Let $f(x)$ be a given function defined on $[a,b]$ then norm of $f(x)$ is given by

$$\|f(x)\| = \left[\int_a^b f^2(x)\, dx\right]^{\frac{1}{2}}$$

Example. Given $g_1(x) = \cos x$, $x \in [-\pi, \pi]$

Find $\|g_1(x)\|$

Solution. Required norm

$$\|g_1(x)\| = \left[\int_{-\pi}^{\pi} \cos^2 x\, dx\right]^{\frac{1}{2}}$$

$$= \left[\int_{-\pi}^{\pi} \frac{1+\cos 2x}{2}\, dx\right]^{\frac{1}{2}} = \left[\frac{1}{2}\, 2\pi\right]^{\frac{1}{2}} = \sqrt{\pi}$$

Orthogonal functions. Let $f_1(x)$ and $g_1(x)$ be any two functions defined on $[a, b]$ then $f_1(x)$ is said to be orthogonal to $g_1(x)$ if

$$\int_a^b f_1(x)\, g_1(x)\, dx = 0$$

i.e Inner product of $f_1(x)$ and $g_1(x)$ is zero.

Example. $f_1(x) = 4$, $g_1(x) = x^3$, $x \in [-2, 2]$

Here $f_1(x)$ is orthogonal to x^3 as

$$\int_{-2}^{2} 4 \cdot x^3\, dx = 0.$$

Orthogonal set of functions. Let $\{f_1(x), f_2(x), \ldots \ldots f_m(x)\}$ be a set of functions defined on $[a, b]$ then This set is said to be orthogonal if

$$\int_a^b f_m(x)\, f_n(x)\, dx = 0 \quad \text{for } m \neq n.$$

Example. The set $\{\sin x, \sin 2x, \sin 3x, \ldots, \sin mx\}$, $x \in [0, \pi]$ is an orthogonal set as

$$\int_0^\pi \sin mx \sin nx\, dx = 0$$

$$LHS = \frac{1}{2}\int_0^\pi 2 \sin mx \sin nx\, dx$$

$$= \frac{1}{2}\int_0^\pi [\cos(m-n)x - \cos(m+n)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^\pi$$

$$= 0.$$

Scanned by CamScanner

Orthonormal set of functions. A set $\{f_1(x), f_2(x), \dots \dots f_m(x)\}$, $x \in [a,b]$ is said to be orthonormal

if
$$\int_a^b \left[\frac{f_m(x)}{\|f_m(x)\|} \cdot \frac{f_n(x)}{\|f_n(x)\|}\right] dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

i.e. $$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = 0$$

where
$$\phi_m(x) = \frac{f_m(x)}{\|f_m(x)\|}$$

and
$$\phi_n(x) = \frac{f_n(x)}{\|f_n(x)\|}$$

Example. Let $\{\sin x, \sin 2x, \sin 3x, \dots \sin mx\}$ be a set on $[0, \pi]$ Then this set is orthonormal set as

$$\int_0^\pi \frac{\sin mx}{\|\sin mx\|} \cdot \frac{\sin nx}{\|\sin nx\|}\, dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

Here
$$\|\sin mx\| = \left[\int_0^\pi \sin^2 mx\, dx\right]^{\frac{1}{2}}$$
$$= \left[\int_0^\pi \left(\frac{1-\cos 2mx}{2}\right) dx\right]^{\frac{1}{2}}$$
$$= \frac{1}{\sqrt{2}} \left[\int_0^\pi dx\right]^{\frac{1}{2}} = \sqrt{\frac{\pi}{2}}$$

Similarly
$$\|\sin nx\| = \sqrt{\frac{\pi}{2}}$$

$$\therefore \int_0^{\pi} \frac{\sin mx}{\|\sin mx\|} \cdot \frac{\sin nx}{\|\sin nx\|}\, dx$$

$$= \int_0^{\pi} \frac{\sin mx}{\sqrt{\frac{\pi}{2}}} \cdot \frac{\sin nx}{\sqrt{\frac{\pi}{2}}}\, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} \sin mx \sin nx \, dx$$

$$= \frac{2}{\pi} \cdot \int_0^{\pi} \frac{1}{2} \cdot 2 \sin mx \sin nx \, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \left[\cos(m-n)x - \cos(m+n)x\right] dx$$

$$= 0 \quad \text{if } m \neq n$$

and

$$\int_0^{\pi} \frac{\sin mx}{\|\sin mx\|} \cdot \frac{\sin nx}{\|\sin nx\|}\, dx$$

$$= \int_0^{\pi} \frac{\sin^2 mx}{\sqrt{\frac{\pi}{2}} \cdot \sqrt{\frac{\pi}{2}}}\, dx \quad \text{if } m = n$$

$$= \frac{2}{\pi} \int_0^{\pi} \left(\frac{1-\cos 2mx}{2}\right) dx \quad \because \sin^2 mx = \frac{1-\cos 2mx}{2}$$

$$= \frac{1}{\pi} \cdot \pi = 1$$

$$\therefore \int_0^{\pi} \frac{\sin mx}{\|\sin mx\|} \frac{\sin nx}{\|\sin nx\|}\, dx = \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases}$$

Therefore, the set $\{\sin x, \sin 2x, \sin 3x, \ldots \sin mx\}$ is orthonormal set on $[0, \pi]$.

Note. One can form orthonormal set if it is orthogonal set on given interval. The mathematician Gram-Schmidt gave this idea so this process of formation of orthonormal set is called Gram-Schmidt process of orthonormalization.

Gram-Schmidt process of orthonormalization

Suppose $\{f_1(x), f_2(x), \ldots \ldots f_n(x)\}$ be a given set of independent real functions on $[a, b]$

To construct $\phi_n(x)$ such that

$$\int_a^b \phi_n(x)\,\phi_m(x)\,dx = \begin{cases} 0, & m \neq n \\ 1, & m = n. \end{cases}$$

choose $f_1(x)$ and construct $\phi_1(x)$ as

$$\phi_1(x) = \frac{f_1(x)}{\|f_1(x)\|} = \frac{g_1(x)}{\|g_1(x)\|} \quad \text{where } g_1(x) = f_1(x).$$

Now, Let

$$g_2(x) = f_2(x) + c_1\phi_1(x) \qquad \text{--- (1)}$$

where c_1 is a constant and c_1 is choosen such that

$$\int_a^b g_2(x)\,\phi_1(x)\,dx = 0$$

$$\Rightarrow \int_a^b \left[f_2(x) + c_1\phi_1(x)\right]\phi_1(x)\,dx = 0 \quad \left[\text{Pick } g_2(x) \text{ from (1)}\right]$$

$$\Rightarrow \int_a^b f_2(x)\,\phi_1(x)\,dx + c_1\int_a^b \phi_1^2(x)\,dx = 0$$

$$\Rightarrow c_1 = -\int_a^b f_2(x)\,\phi_1(x)\,dx \quad \left[\because \int_a^b \phi_1^2(x)\,dx = 1\right]$$

Now, $\phi_2(x)$ can be constructed as

$$\phi_2(x) = \frac{g_2(x)}{\|g_2(x)\|}$$

Again, let

$$g_3(x) = f_3(x) + c_1\phi_1(x) + c_2\phi_2(x) \quad \text{--- (2)}$$

where c_1 and c_2 are constants these constants can be choosen such that

$$\int_a^b g_3(x)\phi_1(x)\,dx = 0 \text{ and } \int_a^b g_3(x)\phi_2(x)\,dx = 0$$

$\because \int_a^b g_3(x)\phi_1(x)\,dx = 0$

$\Rightarrow \int_a^b [f_3(x) + c_1\phi_1(x) + c_2\phi_2(x)]\phi_1(x)\,dx = 0$, from (2)

$\Rightarrow \int_a^b f_3(x)\phi_1(x)\,dx + c_1\int_a^b \phi_1^2(x)\,dx + c_2\int_a^b \phi_2(x)\phi_1(x)\,dx = 0$

$\Rightarrow \int_a^b f_3(x)\phi_1(x)\,dx + c_1\int_a^b \phi_1^2(x)\,dx + 0 = 0$

$\Rightarrow c_1 = -\int_a^b f_3(x)\phi_1(x)\,dx$ as $\int_a^b \phi_1^2(x)\,dx = 1$

and

$\int_a^b g_3(x)\phi_2(x)\,dx = 0 \Rightarrow$

$\int_a^b [f_3(x) + c_1\phi_1(x) + c_2\phi_2(x)]\phi_2(x)\,dx = 0$, from (2)

$\Rightarrow \int_a^b f_3(x)\phi_2(x)\,dx + c_1\int_a^b \phi_1(x)\phi_2(x)\,dx + c_2\int_a^b \phi_2^2(x)\,dx = 0$

$\Rightarrow \int_a^b f_3(x)\phi_2(x)\,dx + 0 + c_2$ $\left[\because \int_a^b \phi_1(x)\phi_2(x)\,dx = 0 \text{ and } \int_a^b \phi_2^2(x)\,dx = 1\right]$

$\Rightarrow c_2 = -\int_a^b f_3(x)\phi_2(x)\,dx$

Putting these values of c_1 & c_2 one can get g_3.

Now we can construct

$$\phi_3(x) = \frac{g_3(x)}{\|g_3(x)\|}$$

. . . .

In general

Let

$$g_n(x) = f_n(x) + c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_{n-1}\phi_{n-1}(x) \quad \text{--- (n)}$$

where the constants $c_1, c_2, \ldots, c_{n-1}$ can be choosen such that

$$\int_a^b g_n(x)\phi_1(x)\,dx = 0 \Rightarrow c_1 = -\int_a^b f_n(x)\phi_1(x)\,dx$$

$$\int_a^b g_n(x)\phi_2(x)\,dx = 0 \Rightarrow c_2 = -\int_a^b f_n(x)\phi_2(x)\,dx.$$

$$\vdots$$

$$\int_a^b g_n(x)\phi_{n-1}(x)\,dx = 0 \Rightarrow c_n = -\int_a^b f_n(x)\phi_{n-1}(x)\,dx$$

after finding $c_1, c_2, \ldots, c_n$ and putting in (n) we can get $g_n(x)$.

Now, construct

$$\phi_n(x) = \frac{g_n(x)}{\|g_n(x)\|}$$

Therefore we have construct the orthonormal set

$$\phi_n(x) = \frac{g_n(x)}{\|g_n(x)\|}, \quad n = 1, 2, 3, \ldots.$$

Example. Given $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = x^2$

$f_4(x) = x^3$, ------ , $x \in]-1, 1[$

find their corresponding orthonormal set.

solution.

Let $g_1(x) = f_1(x) = 1$

then $\phi_1(x) = \dfrac{g_1(x)}{\|g_1(x)\|} = \dfrac{1}{\|1\|}$ — (1)

$$\|1\| = \left[\int_{-1}^{1} 1 \cdot dx\right]^{\frac{1}{2}} = \sqrt{[x]_{-1}^{1}} = \sqrt{2}$$

$\therefore \dfrac{g_1(x)}{\|g_1(x)\|} = \dfrac{1}{\|1\|} \Rightarrow \phi_1(x) = \dfrac{1}{\sqrt{2}}$

Let

$g_2(x) = f_2(x) + c_1\phi_1(x)$ — (2)

where c_1 is so choosen so that

$$\int_{-1}^{1} g_2(x)\phi_1(x)\,dx = 0$$

$$\Rightarrow \int_{-1}^{1}[f_2(x) + c_1\phi_1(x)]\phi_1(x)\,dx = 0$$

$$\Rightarrow c_1 = -\int_{-1}^{1} f_2(x)\phi_1(x)\,dx$$

$\Rightarrow c_1 = -\int_{-1}^{1} x \cdot \frac{1}{\sqrt{2}}\,dx \Rightarrow c_1 = -\frac{1}{\sqrt{2}} \cdot \frac{1}{2}[x^2]_{-1}^{1}$

$= 0$

$\therefore g_2(x) = f_2(x)$

and so $\phi_2(x) = \dfrac{g_2(x)}{\|g_2(x)\|} = \dfrac{x}{\|x\|}$, $\|x\| = \left[\int_{-1}^{1} x^2\,dx\right]^{\frac{1}{2}}$

$\Rightarrow \|x\| = \sqrt{\frac{2}{3}}$

$\therefore \phi_2(x) = \dfrac{x}{\sqrt{2}} \times \sqrt{3} \Rightarrow \phi_2(x) = \sqrt{\dfrac{3}{2}} \cdot x$

Let

$$g_3(x) = f_3(x) + c_1\phi_1(x) + c_2\phi_2(x)$$

where c_1 and c_2 are choosen such that

$$\int_{-1}^{1} g_3(x)\phi_1(x)\,dx = 0 \Rightarrow c_1 = -\int_{-1}^{1} f_3(x)\phi_1(x)\,dx$$

and

$$\int_{-1}^{1} g_3(x)\,\phi_2(x)\,dx = 0 \Rightarrow c_2 = -\int_{-1}^{1} f_3(x)\,\phi_2(x)\,dx$$

$$\therefore\ c_1 = -\int_{-1}^{1} x^2 \cdot \frac{1}{\sqrt{2}}\,dx = -\frac{1}{\sqrt{2}} \cdot \frac{1}{3} \cdot 2$$

$$\Rightarrow c_1 = -\frac{\sqrt{2}}{3}$$

and

$$c_2 = -\int_{-1}^{1} x^2 \cdot \sqrt{\frac{3}{2}}\,x\,dx = 0$$

$$\therefore\ g_3(x) = x^2 + \left(-\frac{\sqrt{2}}{3}\right) \cdot \frac{1}{\sqrt{2}} + 0$$

$$\Rightarrow g_3(x) = x^2 - \frac{1}{3} = \frac{1}{3}\left[3x^2 - 1\right]$$

$$\|g_3(x)\| = \left[\int_{-1}^{1} g_3^2(x)\,dx\right]^{\frac{1}{2}} = \left[\int_{-1}^{1} \frac{1}{9}(3x^2-1)^2\,dx\right]^{\frac{1}{2}}$$

$$= \frac{1}{3}\left[\int_{-1}^{1}(9x^4 - 6x^2 + 1)\,dx\right]^{\frac{1}{2}}$$

$$= \frac{1}{3}\left[\frac{18}{5} - \frac{12}{3} + 2\right]^{\frac{1}{2}} = \frac{1}{3}\left[\frac{18}{5} - 2\right]^{\frac{1}{2}}$$

$$= \frac{1}{3} \cdot \sqrt{\frac{8}{5}} = \frac{2\sqrt{2}}{3\sqrt{5}}$$

$$\therefore\ \phi_3(x) = \frac{g_3(x)}{\|g_3(x)\|} = \frac{(x^2 - \frac{1}{3})}{2\sqrt{2}} \cdot 3\sqrt{5} = \left(\frac{3x^2-1}{2\sqrt{2}} \times \sqrt{5}\right)$$

Similarly we can find $\phi_4(x)$, ... and so on.

(11) Dr. Saleh Hr

Some Questions for Practice

1. Show that the function $g_m(x) = \cos mx$, $m = 0, 1, 2, \ldots$ form orthogonal set of functions on $[-\pi, \pi]$ and obtain their corresponding orthonormal set of functions.

Hint. show $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0$ for $m \neq n$.

and $\displaystyle \phi_0(x) = \frac{g_0(x)}{\|g_0(x)\|} = \frac{1}{\|1\|}$, $\displaystyle \|1\| = \left[\int_{-\pi}^{\pi} 1 \, dx\right]^{\frac{1}{2}}$

$$\phi_m(x) = \frac{g_m(x)}{\|g_m(x)\|} = \frac{\cos mx}{\|\cos mx\|}, \quad m = 1, 2, 3, \ldots$$

$m = 1, 2, 3, \ldots$

$$\|\cos mx\| = \left[\int_{-\pi}^{\pi} \cos^2 mx \, dx\right]^{\frac{1}{2}}, \quad m = 1, 2, 3, \ldots$$

Note: $\|\cos x\| = \|\cos 2x\| = \|\cos 3x\| = \cdots\cdots$

(2) Show that the set $\left\{ \sin\left(\frac{n\pi x}{c}\right), n = 1, 2, 3, \ldots \right\}$ is an orthogonal set on $]0, c[$ and find their orthonormal set.

Hint. Show $\displaystyle\int_0^c \sin \frac{n\pi x}{c} \sin \frac{m\pi x}{c} \, dx = 0$ for $m \neq n$

and get $\displaystyle\frac{g_n(x)}{\|g_n(x)\|}$ where $\displaystyle g_n(x) = \frac{\sin\left(\frac{n\pi x}{c}\right)}{1}$, $n = 1, 2, 3, \ldots$

(3) Show that the functions $1, \cos 2x, \cos 4x, \ldots$ are orthogonal on $[0, \pi]$ and find their corresponding orthonormal set.

Hint. show $\int_0^{\pi} \cos 2nx \cos 2mx \, dx = 0$ for $m \neq n$

and find $\phi_0(x) = \frac{1}{\|1\|}$, $\|1\| = \left[\int_0^{\pi} 1 . dx\right]^{\frac{1}{2}}$

$$\phi_n(x) = \frac{\cos 2nx}{\|\cos 2nx\|}, \quad n = 1, 2, 3, \ldots$$

$$, \|\cos 2nx\| = \left[\int_0^{\pi} \cos^2 2nx \, dx\right]^{\frac{1}{2}}, \quad n = 1, 2, 3, \ldots.$$

(4). Show that the function $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on $]-1, 1[$ and find the constant A and B so that $f_3(x) = (1 + Ax + Bx^2)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ on $]-1, 1[$.

Hint. show $\int_{-1}^{1} f_1(x) f_2(x) \, dx = 0$

and to calculate A and B, take

$$\int_{-1}^{1} f_3(x) f_1(x) \, dx = 0$$

and $\int_{-1}^{1} f_3(x) f_2(x) \, dx = 0$

and solve both equations to get A and B.

(5). Show that the functions $1, \cos\frac{2n\pi x}{T}, \sin\frac{2n\pi x}{T}$, $n = 1, 2, 3, \ldots$ are orthogonal on $[-\frac{T}{2}, \frac{T}{2}]$ and find their corresponding orthonormal set.

Hint. ~~Show~~ Let $f_1(x) = 1$

$$f_2(x) = \cos\frac{2n\pi x}{T}$$

$$f_3(x) = \sin\ldots$$

then show

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(x) f_2(x)\, dx = 0, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} f_2(x) f_3(x)\, dx = 0,$$

and

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(x) f_3(x)\, dx = 0$$

and get

$$\frac{f_1(x)}{\|f_1(x)\|}, \quad \frac{f_2(x)}{\|f_2(x)\|}, \quad \frac{f_3(x)}{\|f_3(x)\|}.$$